CP$_N$ solitons in quantum Hall systems

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Abstract. We present here an elementary pedagogical introduction to CP_N solitons in quantum Hall systems. We begin with a brief introduction to both CP_N models and to quantum Hall (QH) physics. We then focus on spin and layer-spin degrees of freedom in QH systems and point out that these are in fact CP_N fields for $N = 1$ and $N = 3$. Excitations in these degrees of freedom will be shown to be topologically non-trivial soliton solutions of the corresponding CP_N field equations. We conclude with a brief summary of our own recent work in this area, done with Sankalpa Ghosh.

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1 Introduction

CP_N quantum fields were introduced in the mid-seventies in the particle physics literature as two-dimensional models which bore important similarities to four-dimensional Quantum Chromodynamics. It was shown that these field theories were very interesting in their own right [1]. Among their important features was the availability of exact solitary wave solutions of prototype CP_N models at the classical level, even though the underlying field equations were coupled non-linear partial differential equations in 2+1 dimensions. These solutions, obtained through elegant methods, could be written in terms of simple analytic functions. They were also "topological solitons", *i.e.* they could be classified into homotopy sectors characterised by a winding number. In real 4-dimensional particle physics these beautiful solutions remained as theoretical discoveries in toy models with no experimental manifestation. However, subsequently they were shown to be physically realisable in an entirely different arena of physics, namely, two dimensional quantum Hall systems.

We present here an elementary pedagogical introduction to CP_N solitons in quantum Hall systems. We begin with a brief introduction to both CP_N models and to quantum Hall (QH) physics. We then focus on spin and layer-spin degrees of freedom in QH systems and point out that these are in fact CP_N fields for $N = 1$ and $N = 3$. Excitations in these degrees of freedom will be shown to be topologically non-trivial soliton solutions of the corresponding CP_N field equations. We conclude with a brief summary of our own recent work in this area, done with Sankalpa Ghosh.

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2 CP$_N$ fields

A CP_N field is a multiplet of $N + 1$ complex fields which are functions of some d-dimensional space-time (which we will denote by x), subject to two conditions we will list below. For the present we leave the space-dimensionality open and later concentrate on the case of 2 dimensions. This field multilplet can be denoted by a CP_N spinor

$$\eta_\sigma(x) = \begin{pmatrix} \eta_1(x) \\ \eta_2(x) \\ ... \\ ... \\ \eta_{N+1}(x) \end{pmatrix}. \qquad (1)$$

To qualify as a CP_N spinor this multiplet has to obey, *at each point x*,

(i) Normalisation:

$$\sum_\sigma |\eta_\sigma(x)|^2 \;=\; 1 \qquad (2)$$

and

(ii) equivalence under local $U(1)$ transformations (gauge invariance):

$$\eta_\sigma(x) \;\approx\; \eta_\sigma(x) e^{i\Lambda(x)} \qquad (3)$$

where $\Lambda(x)$ can be an arbitrary real function of x, but the same for all the components σ. Altogether then there are $2(N + 1) - 2 \;=\; 2N$ real degrees of freedom at each x.

The system could have any action functional and field equations as long as they are gauge invariant/covariant under the $U(1)$ transformations above. Now, typically, field equations involve gradients of fields. But under the gauge transformations (3) gradients are not covariant:

$$\boldsymbol{\nabla}\eta_\sigma(x) \;\to\; e^{i\Lambda(x)}\big(\boldsymbol{\nabla}\eta_\sigma(x) \;+\; i(\boldsymbol{\nabla}\Lambda)\eta_\sigma(x)\big). \qquad (4)$$

However, consider the "covariant derivative"

$$\mathbf{D}\eta_\sigma(x) \equiv \left(\boldsymbol{\nabla} + \mathrm{i}\mathbf{A}\right)\eta_\sigma(x) \tag{5}$$

where

$$\mathbf{A}(x) \equiv \mathrm{i}\sum(\eta_\sigma(x))^*\boldsymbol{\nabla}\eta_\sigma(x). \tag{6}$$

One can check that $\mathbf{A}(x)$ is real and behaves under the gauge transformations as

$$\mathbf{A} \rightarrow \mathbf{A} - \boldsymbol{\nabla}\Lambda. \tag{7}$$

Hence

$$\mathbf{D}\eta_\sigma(x) \rightarrow \mathrm{e}^{\mathrm{i}\Lambda(x)}\mathbf{D}\eta_\sigma(x). \tag{8}$$

Using this, we can construct the simplest prototype CP_N energy functional for static configurations

$$E_{pro}\left[\eta_\sigma(x)\right] = (1/2)\int \mathrm{d}\boldsymbol{x}\sum_\sigma\left(\mathbf{D}\eta_\sigma(x)^*\right)\cdot\left(\mathbf{D}\eta_\sigma(x)\right) \tag{9}$$

yielding coupled non-linear field equations

$$\mathbf{D}\cdot\mathbf{D}\eta_\sigma(x) + \kappa\eta_\sigma(x) = 0 \tag{10}$$

where κ is a Lagrange multiplier implementing the normalisation condition in equation (2). This can be viewed as an equation for static (time-independent) fields in some d-dimensions. Similarly, in Minkowskian $d+1$ dimensions, a CP_N field equation would be

$$\left(D_0^2 - \mathbf{D}\cdot\mathbf{D}\right)\eta_\sigma(x) + \kappa\eta_\sigma(x) = 0 \tag{11}$$

where $D_0 = \partial_t + \mathrm{i}A_0$.

Equations (9, 10) are the simplest rotationally covariant candidates for the energy functional and the field equation, respectively, for CP_N systems. We may call them the prototype CP_N system. Of course any other field equation and energy functional for $N+1$ complex fields would also define a CP_N system, as long as they are covariant under the gauge transformations (3) and consistent with the normalistation constraint (2). Indeed the the CP_N systems that appear in QH physics do have more complicated expressions in their energy and field equations, although they all include the basic prototype terms above.

3 Topological solitons in 2 dimensions

Although the prototype CP_N field equation equation (10) is a set of coupled nonlinear partial diffential equations, an infinite number of exact solutions have been obtained for them in 2 dimensions. These solutions are furthermore topological solitons. We will briefly describe them. The rest of our discussion in this article will be limited to

two space dimensions. As a first step note that the lowest (zero) energy solutions of equation (10) are the gauge equivalent family of spinors

$$\eta_\sigma(\mathbf{r}) = b_\sigma\mathrm{e}^{\mathrm{i}\Lambda(\mathbf{r})} \tag{12}$$

where b_σ is any constant (space independent) CP_N spinor and where the phase factor $\mathrm{e}^{\mathrm{i}\Lambda(\mathbf{r})}$ could be any single valued function. To see this first consider the constant solution (where $\Lambda = 0$). Then since $\boldsymbol{\nabla}b_\sigma = 0$, the vector potential as defined in (6) is also zero. Hence $\mathbf{D}\,b_\sigma = 0$ and $E_{pro}[b_\sigma] = 0$. By gauge invariance of the energy, all members of the gauge class in (12) will also have zero energy.

Turning to configurations of non-zero but finite energy, they must asymptotically (as $r \rightarrow \infty$) tend to this zero energy solution:

$$\eta_\sigma(\mathbf{r}) \rightarrow b_\sigma\mathrm{e}^{\mathrm{i}\Lambda(\theta)} \tag{13}$$

where θ is the angular coordinate on the plane. Note that the angular gradient of such configurations behaves asymptotically as

$$\nabla_\theta\eta_\sigma(\mathbf{r}) \rightarrow \frac{1}{r}\,\partial_\theta\Lambda\,\eta_\sigma(\mathbf{r}) \tag{14}$$

which is not square integrable. However, it is not the plain gradient which occurs in the energy functional (9), but the covariant gradient, which does vanish sufficiently fast asymptotically for the energy integral to have a finite value.

Thus any finite energy configuration corresponds to a particular function $\Lambda(\theta)$ on the circle at spatial infinity. This function is clearly a mapping of a circle into a circle and can be classified by a winding number (the first homotopy group $\Pi_1[S_1]$ is the group of integers). An explicit expression for this winding number in terms of the asymptotic behavior (13) is

$$n = \frac{1}{2\pi}\int \mathrm{d}\theta\frac{\mathrm{d}\Lambda}{\mathrm{d}\theta}$$

$$= \frac{1}{2\pi}\int \mathrm{d}^2r\epsilon_{\mu\nu}(D_\mu\eta_\sigma(x))^*\,(D_\nu\eta_\sigma(x)). \tag{15}$$

Exact solutions are available analytically in every topological sector (*i.e.* in each class of configurations characterised by a given value of the winding number.) It can be derived that

$$\eta_\sigma(z) = K(z)\begin{pmatrix}1\\w_2(z)\\w_3(z)\\...\\...\\w_{N+1}(z)\end{pmatrix} \tag{16}$$

is an exact solution of the field equation (10), where $z = x + \mathrm{i}y$, $\{w_\sigma(z)\}$ are arbitrary analytic functions of z, and

$K(z)$ is the normalisation factor. For example, it can be checked that

$$\eta_\sigma(z) = \frac{1}{\sqrt{a^2 + Nr^{2n}}} \begin{pmatrix} a \\ z^n \\ z^n \\ \cdots \\ \cdots \\ z^n \end{pmatrix} \qquad (17)$$

where a is any constant, is an exact solution. As $r \to \infty$, it behaves as

$$\frac{1}{\sqrt{N}} e^{in\theta} \begin{pmatrix} 0 \\ 1 \\ 1 \\ \cdots \\ \cdots \\ 1 \end{pmatrix} \qquad (18)$$

and clearly has a winding number n in its phase.

These exact solutions are for the prototype CP_N model (10). A realistic physical system describable by a CP_N field will in general have a more complicated energy functional and field equation. But for most such physical systems such as those which appear in the quantum Hall phenomena, the lowest energy solution is still a space-independent spinor. Therefore localised finite energy solitons will still obey the asymptotic condition (13) and be characterised by the same winding number. Lastly, while we have presented here only static solutions in 2 space dimensions, time-dependent moving soliton solutions of equation (11) can be obtained by boosting.

For a more detailed review of CP_N solitons see reference [2].

4 Quantum Hall systems

Since our subject deals with CP_N solitons in Quantum Hall (QH) systems, we would like to give some sort of an overall introduction to the latter for those who may need it. This is a vast subject. Further, the basic phenomena referred to as the Quantum Hall effect (QH) are widely known. Therefore, even though we will begin from the beginning, our overview will be channelised to focus only on those of aspects this system which form pre-requisites to understanding its CP_N excitations.

Recall the classical Hall problem of electrons moving in the xy-plane confined in the $\hat{y}$ direction by boundaries, and in the presence of crossed electric and magnetic fields $\mathbf{E} = E_x\hat{x}$ and $\mathbf{B} = -B\hat{z}$ respectively. As the electrons begin to move in the x-direction because of E_x, the Lorentz force due to the magnetic field will push the electrons towards the y-boundary where they accumulate and produce a transverse electric field $E_y\hat{y}$ which eventually balances the Lorentz force vB/c. As a result the electrons end up moving purely along the $\hat{x}$ direction although the total electric field is $\mathbf{E} = E_x\hat{x} + E_y\hat{y}$. The electric current can be written as $\mathbf{j} = \underline{\sigma} \cdot \mathbf{E}$ where $\underline{\sigma}$ is the conductivity matrix which can be easily calculated using the Drude formulae. Its diagonal elements are $\sigma_{xx} = \sigma_{yy} = ne^2\tau/\mu$ (where μ is the electron

mass, τ is the collision time, and n is the electron density). Its off-diagonal element, the Hall conductivity, is given by $\sigma_{xy} = nec/B$. In terms of the "filling factor" ν defined as the ratio of the density of electrons to fluxons,

$$\nu \equiv \frac{n}{B/\phi_0} \qquad (19)$$

where $\phi_0 = hc/e$ is the unit of flux, the Hall conductivity can be written as $\sigma_{xy} = (e^2/h)\nu$. These expressions for the conductivity tensor were obtained for the simplest possible situation, that of non-interacting classical planar electrons in a perpendicular magnetic field. Electrons in real macroscopic experimental samples are much more complicated. They interact with one another, with the ions in their environment, and obey the rules of many-body quantum mechanics. One would expect the behaviour of their conductivity to be quite complicated and messy in general, as compared to the simple results above. But when Hall effect experiments were done on exceptionally pure samples of 2D electron gas at very low temperatures and very high magnetic fields, it was found that the Hall conductivity σ_{xy} as a function of the filling fraction ν revealed a startlingly simple pattern. It contained, as a function of ν, a series of extraordinarily flat plateaus, with a flatness accurate to better than 1 in 10^7. These plateaus were first found to occur at integer filling fractions with Hall conductance values quantized to be the same integer in units of e^2/h. Furthermore, at those filling fractions where σ_{xy} had plateaus, the diagonal resistivity ρ_{xx} was found to be zero. These phenomena were called the Integer QH Effect (IQHE). Subsequently, in experiments involving higher magnetic fields and higher mobility samples, the same phenomenon of plateaus in σ_{xy} and of vanishing of ρ_{xx} was also found at fractional values of the filling factor. These fractions (with one exception, still only partially understood) corresponded to odd integers in their denominator. This is often called the Fractional QHE (FQHE).

We will concentrate here on the particular case of $\nu = 1$ where the CP_N solitons of interest to us appear. Fortunately, this is also the value of ν at which the physics of QH effect is most easily understood. Let us again start with non-interacting electrons in a transverse uniform B field, but now we treat it quantum mechanically and worry about the effect of interactions later. This problem, solved fully and exactly long ago by Landau, is a now a standard textbook problem in quantum mechanics (see for example [3]). The results in brief are as follows. The system can be mapped into a pair of harmonic oscillators, one with frequency zero and the other with $\omega = \omega_c \equiv eB/\mu c$. Excitations of the latter lead to energy levels

$$E_n = (n + 1/2)\hbar\omega_c. \qquad (20)$$

These are the famous Landau levels. Each level is highly degenerate, corresponding to excitations of the other oscillator which has zero-frequency. The degenerate states lying in the lowest Landau level (LLL) have wavefunctions (in the symmetric gauge) of the form

$$\phi_m = z^m e^{-|z|^2/4l^2} \qquad (21)$$

where $z = x + iy$, $l^2 = \hbar c/eB$ and the integer m ranges from zero to infinity.

The degeneracy is formally infinite for an infinite plane, but on a finite sample it can be shown to be equal to the number of fluxons $N_{deg} = BA/\phi_0$ where A is the area of the sample. Therefore when the filling fraction as defined in (19) is unity, the total number N of electrons exactly equals the number of states in the LLL. Consequently, at unit filling, the ground state of the system will correspond to electrons occupying all the states of the LLL and leaving all higher Landau levels empty. The system clearly has an energy gap equal to $\hbar\omega_c = \hbar eB/\mu c$, which is very large for large magnetic fields. (It is this large gap and the associated incompressibility that is responsible for the occurance of the Hall plateaus at $\nu = 1$. But we do not digress to present that explanation here since we would like to rapidly progress towards CP_N solitons.)

Next consider the wavefunction of that $\nu = 1$ many-body ground state, still staying within the non-interacting approximation. It will be a Slater determinant of all the one-electron states in the LLL given in (21). Apart from the Gaussian factor in each state, this is a determinant of polynomials in z, which is just the Vandermonde determinant and can be rewritten in the Jastrow form. Hence

$$\Psi_{\nu=1} \;=\; \Pi_{i<j}\,(z_i - z_j)\,\exp\left(-\sum_i |z_i|^2/4l^2\right). \quad (22)$$

This is the famous Laughlin wavefunction for the $\nu = 1$ ground state. We have derived it only in the non-interacting approximation. But Laughlin has suggested that this wavefunction would be a very good approximation even when interactions of the electrons with one another and with impurities are taken into account. The reason for this is that this wavefunction already possesses many of the desired features of the exact wavefunction. It is antisymmetric, as required by the Pauli principle. It is an eigenfunction of the total angular momentum, as befits the ground state of a system which is circularly symmetric. It vanishes whenever two electrons coincide — a feature that will reduce all pairwise Coulomb energies. Finally, in the presence of interactions, one would expect the ground state to contain some admixture also of states fronm the higher Landau levels. But in the limit of very large magnetic fields, the energy gap $e\hbar B/\mu c$ is so large that such admixture will be small. These arguments suggest that the Laughlin wavefunction will be robust even in the presence of interactions. Indeed, the Laughlin wavefunction has been found to be in excellent agreement with numerical calculations.

5 Spin excitations

The arguments in the preceding section were incomplete in that the spin degrees of freedom were not considered in the discussion. Even though the electrons are treated as two-dimensional with respect to their coordinates, they are physical 3-dimensional electrons and do carry spin. The spin part of the wavefunction has to be specified. Now,

the Pauli principle requires antisymmetry of the entire wavefunction including spin. But the Laughlin wavefunction (22), which seems to be a very accurate approximation to the correct wavefunction, is already antisymmetric in the coordinates z_i. Therefore the spin part, suppressed in equation (22), must be fully symmetric. That is, all the electron spins must be polarised in the same direction. Thus the QH ground state at $\nu = 1$ is a ferromagnet for the same reason that usual magnets are, *viz.*, to minimise the exchange Coulomb energy. Given that there is a magnetic field along the z-direction, one expects the polarisation to be in the same direction.

Although the magnetic field is very strong, its coupling to the spins is not prohibitively large because of the effective g-factor for electrons is reduced in the layer sandwiched between the two semiconductors. As compared to the value of 2 for free electrons in vacuum, it can be as low as 0.4 here. Of course that is sufficient to align all spins along the B field in the ground state, but excited states are possible at reasonably low energy where some spins point away from the z-direction. Indeed one would expect that the low energy excitations of the system can be described solely by various spin textures, with the coordinate part of the wavefunction still remaining in the LLL since any admixture with higher Landau levels will be very expensive energetically.

Thus the low energy dynamics of the $\nu = 1$ system can be studied by going to the continuum limit and treating the system as a two-dimensional field of unit vectors $\mathbf{m}(\mathbf{x})$ at each point, describing the direction of the spin at that point. This field of unit vectors has a long history under the name of the nonlinear $O(3)$ model. But it is also just a CP_1 field (see [2]). Given a general CP_1 spinor denoted by $\eta_\sigma(\mathbf{r}) \;=\; \begin{pmatrix} \alpha(\mathbf{r}) \\ \beta(\mathbf{r}) \end{pmatrix}$, the quantity $\mathbf{m} \equiv \langle\eta|\boldsymbol{\sigma}|\eta\rangle$, where σ_i are Pauli matrices, will be a unit vector. Thus a CP_1 field is also a unit vector field. The homotopy classification for CP_N discussed in Section 2 can also be recast, for $N = 1$, in terms of the unit vector field $\mathbf{m}(\mathbf{r})$. The boundary condition (13) on $\eta_\sigma(\mathbf{r})$ corresponds to having the unit vector $\mathbf{m}$ take the *same* value everywhere on the boundary of two-dimensional space, which can be therefore compactified into a 2-sphere. Hence any such field configuration $\mathbf{m}(\mathbf{r})$ is a mapping of the 2-sphere S_2 in coordinate space into the 2-sphere of spin directions. These mappings $S_2 \to S_2$ are again classified by a winding number. These configurations are two-dimensional analogues of four-dimensional configurations studied long ago by Skyrme (see [2] for references), and are called skyrmions.

As we have already mentioned exact solutions in all topological classes are available for prototype CP_N models in two space dimensions for all N. In the case of the nonlinear $O(3)$ model, its skyrmion solutions had already been discovered by Belavin and Polyakov before its generalisation to CP_N models had been developed [4]. Turning to QH systems, Sondhi *et al.* [5], in a very interesting paper, showed that not only can these exotic skyrmion excitations occur at $\nu = 1$, but that they are in fact the lowest energy excitations, lower in energy than single

spin-flips. Subsequently experimental support for the existence of such skyrmionic (CP_1) excitations was also found [6].

6 Psuedo(layer)spin in QH systems

Following the spectacular quantum Hall results for electrons in a layer, more complicated experiments were done using samples that contained two parallel layers of electrons [7]. Some more interesting results emerged. One would expect of course to see results where the system behaves as a simple additive sum of each layer. Thus one would expect to see quantum Hall plateaus at total filling, in both layers together, of $\nu_{total} = 2$ or $\nu_{total} = 2/3$ corresponding to the observed single layer plateaus at ν equal to 1 and 1/3 respectively. Indeed this is what is seen when the layer separation d is large. But when d is reduced to about $3l$, quantum Hall plateaus appear at *total* filling ν_{total} equal to unity. The electrostatic capacitance energy between the two layers would require them to have equal densities of electrons which corresponds to a filling of 1/2 in each layer. But no quantum Hall effect is seen in monolayers at $\nu = 1/2$.

Therefore this plateau at $\nu_{total} = 1$ clearly cannot be understood by thinking of the system as a pair of independent layers. Rather, the phenomenon must reflect some sort of a quantum coherence between the two layers. An ingenious formulation for understanding this, developed by Girvin, Macdonald and co-workers is to associate a normalised 2-component layer-spin or pseudospin $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ to each electron [8]. These components α and β give the amplitude for the electron being in the upper and lower layers respectively. Let us for the moment suppress real spin and see what the occurrence of the plateau means for the pseudospin. As we mentioned earlier, a $\nu = 1$ Hall plateau is well described by the antisymmetric Laughlin wavefunction (22). If this should also hold for the double layer, then the Pauli principle requires that the pseudospin of the electrons should be fully symmetric, *i.e.* the system must be a pseudospin ferromagnet similar to (and for the same exchange-energy-reducing reasons as) the real spin ferromagnetism. Further, if there is even the smallest tunnelling probability between the two layers, the ground state will be a symmetric superposition of the two layers, *i.e.*, be in the pseudospin state $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$. In other words the pseudospin magnet will point along the x-direction. This again is similar to the real spin magnet pointing along the z-axis because of its Zeeman coupling to the magnetic field. The tunnelling term in the Hamiltonian will act as the analogue of the Zeeman coupling for the pseudospin.

With the ground state being a ferromagnet in layer space, once again the system will carry low energy excitations corresponding to different peudospin textures. Asymptotically they will have to go to the ground state value, along the x-direction, but in the interior one could have any smooth configuration of direction vectors. Once

again we have an $O(3)$ or CP_1 field in compactified 2-space, giving rise to a topological classification of all solutions by a winding number. Such topological solutions, called bi-merons in this context, were first discussed in detail by Moon *et al.* [9]. Ghosh and I have also studied these solutions and evaluated their detailed profiles and energies [10].

7 Spin-pseudospin intertwined CP_3 solitons

Now let us add on spin degrees of freedom to the preceding discussion of double layer systems. The full Hall fluid ground state at $\nu_{total} = 1$ will be ferromagnetic in both spin and pseudospin, with a coordinate dependence given by the Laughlin wavefunction. The combined spin and pseudospin part of the wavefunction can be described by a 4-component multiplet:

$$\eta_\sigma(x) \quad = \begin{pmatrix} \eta_1(x) \\ \eta_2(x) \\ \eta_3(x) \\ \eta_4(x) \end{pmatrix} \tag{23}$$

where the spin-pseudospin index $\sigma = 1, 2, 3, 4$ corresponds to amplitudes that the electron is in the upper-layer up-spin, upper-layer down-spin, lower-layer up-spin and lower-layer down-spin states respectively. Such 4-components spinors were first studied in QH systems by Arovas *et al.* [11] and by Ezawa [12]. Since these probabilites must add up to one, the spinor has to be normalised and looks like a CP_3 spinor. But that requires the further restriction that the spinor be defined only modulo a local gauge transformation common to all four components. This in turn requires that the energy functional of the spinor field enjoy a corresponding gauge invariance. Ghosh and I [13] ensured that this was so by calculating the energy of this 4-component field starting from the microscopic Hamiltonian, following the procedure used by the Indiana group [8,9] for the purely pseudospin case. Let us summarise how this is done. We work in the second quantised formalism in terms of the 4-component electron field $\psi_\sigma^\dagger(\mathbf{r})$.

The microscopic Hamiltonian is

$$H = \sum_{\sigma,\delta} \int d\mathbf{r} \psi_\sigma^\dagger(\mathbf{r}) \left(\tilde{g}\hat{\sigma}_z - t\hat{\tau}_x \right)_{\sigma\delta} \psi_\delta(\mathbf{r})$$

$$+ \frac{1}{2} \sum_{\sigma_1,\sigma_2=1}^{4} \int d\mathbf{r}_1 d\mathbf{r}_2 \psi_{\sigma_1}^\dagger(\mathbf{r}_1)\psi_{\sigma_2}^\dagger(\mathbf{r}_2)$$

$$\times V^{\sigma_1\sigma_2}(\mathbf{r}_1 - \mathbf{r}_2)\psi_{\sigma_2}(\mathbf{r}_2)\psi_{\sigma_1}(\mathbf{r}_1). \tag{24}$$

In the above, the Coulomb potential $V^{\sigma_1\sigma_2}$ depends on whether the particles are in the same layer or different layers, $\hat{\sigma}_z$ and $\hat{\tau}_x$ are spin and pseudospin matrices suitably generalised as 4×4 matrices on the outer product space of spin and pseudospin, and $\tilde{g}$ and t are the Zeeman and tunnelling couplings. (The kinetic term (in the presence of the B field) can be suppressed for our purposes since our excitations involve only LLL states all of which carry

the same constant energy of $\hbar\omega_c/2$.) The field-theoretic state vector corresponding to any given spin-pseudospin texture η can be written as

$$| \Psi \rangle = \prod_X \left[\sum_\sigma C_{\sigma X}^\dagger \eta_\sigma(X) \right] | 0 \rangle \qquad (25)$$

where $| 0 \rangle$ is the vacuum (no electron) state, X stands for Landau gauge orbitals and $\eta_\sigma(X)$ is an orbital-dependent 4-spinor. The energy functional for a given spin-pseudospin texture is then obtained to leading order by evaluating the expectation value of the Hamiltonian (24) in the state (25). The result, upon following the same steps as pionerered by the Indiana group [8,9], is

$$
\begin{aligned}
E[a_\sigma] = \frac{1}{2\pi l^2} & \int d\mathbf{r} \Big[\tilde{g} \left(|a_1|^2 - |a_2|^2 + |a_3|^2 - |a_4|^2 \right) \\
& - t \left(a_1 a_3^* + a_2 a_4^* + \text{h.c.} \right) \Big] \\
+ \beta & \int d\mathbf{r} (|a_1(X)|^2 + |a_2(X)|^2 - |a_3(X)|^2 - |a_4(X)|^2)^2 \\
+ 2\rho^s & \int d\mathbf{r} \left[\sum_{i=1,4} (\partial_\mu a^{i*}(\mathbf{r}) \partial^\mu a^i(\mathbf{r})) + \sum_{i=1,4} a^{i*}(\mathbf{r}) \partial_\mu a^i(\mathbf{r})^2 \right] \\
+ (\rho^d - \rho^s) & \int d\mathbf{r} \Big[a^1 a^{3*} \boldsymbol{\nabla}^2(a^3 a^{1*}) + a^1 a^{4*} \boldsymbol{\nabla}^2(a^4 a^{1*}) \\
& + a^2 a^{3*} \boldsymbol{\nabla}^2(a^3 a^{2*}) + a^2 a^{4*} \boldsymbol{\nabla}^2(a^4 a^{2*}) + \text{h.c.} \Big] \qquad (26)
\end{aligned}
$$

where the constants β, ρ^d and ρ^s are calculated from the direct and exchange Coulomb energies.

Note that the third term in the energy functional (26) is just the protoype CP_3 energy in (9). Our full expression for the energy is more complicated. It can however be noticed that all the other terms are also gauge invariant under the $U(1)$ transformations (3). Therefore we *are* dealing with a CP_3 theory. All the general discussion given earlier for CP_N theories apply. Topological soliton solutions can be obtained for the field equations which in turn can be derived by extremising the energy (26).

Explicit soliton solutions have been obtained by us [13], by numerically solving the coupled nonlinear partial differential equations that arise when (26) is extremised. In particular we concentrated on interesting new topological CP_3 solitons where the spin and pseudospin intertwine non-trivially. For example, the very simple texture

$$A \begin{pmatrix} \lambda \\ z - b \\ 0 \\ z + b \end{pmatrix} \qquad (27)$$

corresponds to a spin-skyrmion in the upper layer and also a "bi-meron" in the layer spin of the down-spin component. This simple ansatz will of course not satisfy the full field equations. But we have obtained numerical solutions with similar intertwined spin-pseudospin topology. Lack of space here does not allow us to describe in detail these solutions. Readers interested in their detailed profile as well as the numerical methods used are referred to [13].

We have also calculated the energy of these solutions and minimised it with respect to the parameters in the ansatz. The resulting cost of creating a pair of such topologically intertwined spin-pseudospin excitations works out to be about $1.2(e^2/\epsilon l)$ as compared to particle hole excitations which cost about $1.25(e^2/\epsilon l)$. That the former energy is a little smaller is not to be taken seriously, given the various approximations that have gone into our energy calculations. All one can say is that it is possible that our topological CP_3 excitations may well be the lowest in energy, but to be sure of this one must make longer and more precise calculations.

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